Analytical expression of the potential generated by a massive inhomogeneous straight segment

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Abstract. Potential calculation is an important task to study dynamical behavior of test particles around celestial bodies. Gravitational potential of irregular bodies is of great importance since the discoveries of binary asteroids, this opened a new field of research. A simple model to describe the motion of a test particle, in that case, is to consider a finite homogeneous straight segment. In our work, we take this model by adding an inhomogeneous distribution of mass. To be consistent with the geometrical shape of the asteroid, we explore a parabolic profile of the density. We establish the closet analytical form of the potential generated by this inhomogeneous massive straight segment. The study of the dynamical behavior is fulfilled by the use of Lagrangian formulation, which allowed us to give some two and three dimensional orbits.
Keywords : Potential-Inhomogeneous distribution-Asteroids.

I- Introduction

The discovery of irregular small bodies and binary asteroids like Ida and Doctyl, gave a rise to the potential calculation. Many attempts have been made to approximate the potential. In [1], Riaguas et al. proposed a homogeneous straight segment. Elipe et al. described in [2] the motions around (433) Eros with the same homogeneous model. A polyhedron and harmonic was used by Werner and Scheeres for asteroid 4769 Castalia in [3] and [4]. Ellipsoids, material points and double material segment was used by Przemyslaw et al in [5] and [6], as the model of irregular elongated bodies. In our work we give a new idea to models the potential generated by an elongated body. We consider a straight massive segment with variable density. To be consistent with the geometrical aspect of the asteroid, we use a parabolic profile. Our work generalize that of Riaguas et al.[1]. In the first part of this work, we establish the closet forme of the potential generated by an inhomogeneous massive straight segment. In the second part we study the dynamical behavior of a test particle in the field of the straight segment. We conclude in the last part by the numerical resolution of the differential equations of motion. In this part we show some orbits in two and three dimension.

II- Potential calculation

We consider an inhomogeneous straight segment of length $2l$ and mass M which lies along the $x - axis$, with a parabolic profile of linear mass density (Fig.1), expressed by

$$\lambda(x) = -ax^2 + b \tag{1}$$

in which a and b are linked by $a < \frac{b}{l^2}$ and $M = -\frac{2}{3}al^3 + 2bl$.

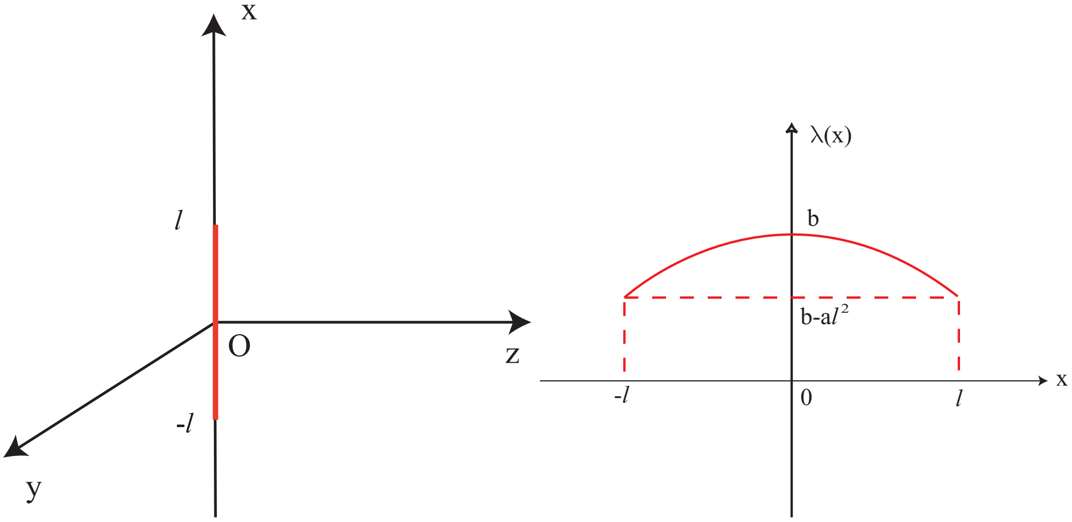

Fig. 1: *Left : Straight segment in reference frame* $(Oxyz)$. *Right : profile of density.*

At a point P, the gravitational potential generated by the segment is :

$$U(P) = -G \int \frac{dm}{r} \tag{2}$$

Where G is the gravitational constant. r is the distance between P and the infinitesimal mass dm located at H with abscissa x_H in the segment. Fig.2.

1. Expression of $HP = r$ (Fig.2) :
   Let us consider an inertial reference frame $(Oxyz)$, and let $\overrightarrow{r}_1$, and $\overrightarrow{r}_2$ be the position vectors of the end points of the straight segment. The position vector of a point of segment is given by

$$\overrightarrow{r} = \overrightarrow{HP} = \overrightarrow{HH_1} + \overrightarrow{H_1P}$$

then

$$\overrightarrow{r} = \overrightarrow{r_1} - \frac{1}{2}\left(1 + \frac{x_H}{l}\right)\overrightarrow{x}_{12}$$

where

$$\overrightarrow{x}_{12} = 2l\overrightarrow{e_x}$$

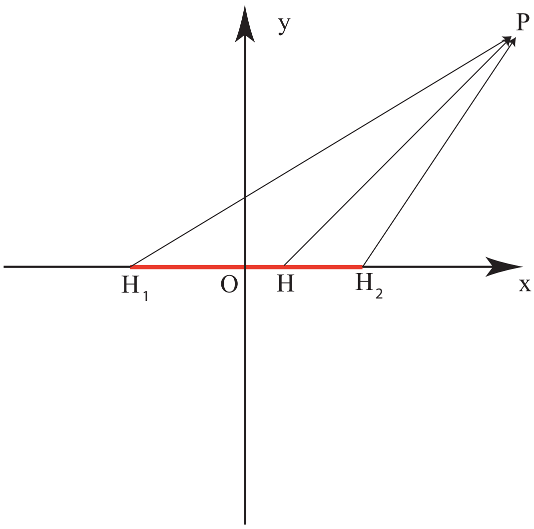

Fig. 2: *Straight segment* .

Next, define

$$\nu = \frac{1}{2}(1 + \frac{x_M}{l})$$

as a new variable of integration with $0 \leq \nu \leq 1$.
After calculation we obtain

$$r^2 = r_1^2 + 4l^2\nu^2 + \nu\left(r_2^2 - r_1^2 - 4l^2\right) \tag{3}$$

2. Expression of dm :
   The infinitesimal mass dm located at H with abscissa x_H is given by

$$dm = \lambda(x_H)dx_H = 2l(-ax_H^2 + b)d\nu$$

Hence

$$dm = 2l(-4al^2\nu^2 + 4al^2\nu + b - al^2)d\nu \tag{4}$$

By substituting (3) and (4) in (2) and developing the calculation we obtain

$$U(r_1, r_2) = 4al^2G\int_0^1 \frac{\nu^2 - \nu - \frac{b-al^2}{4al^2}}{\sqrt{\nu^2 + \nu\left(\frac{r_2^2-r_1^2-4l^2}{4l^2}\right) + \frac{r_1^2}{4l^2}}}d\nu \tag{5}$$

After some laborious calculation and simplification we achieve the closet expression of the potential generated at P :

$$U(r_1, r_2) = \frac{G}{32l^2}\left\{\begin{array}{l} 16al^3\left(r_2 + r_1\right) + 12al\left(r_2 - r_1\right)\left(r_1^2 - r_2^2\right) + \\ \left[\begin{array}{l} 8al^2\left(r_2 + r_1\right)^2 - 16al^2r_1r_2 \\ -3a\left(r_1 - r_2\right)^2\left(r_2 + r_1\right)^2 - 16al^4 + 32bl^2 \end{array}\right] \ln\left(\frac{r_2 + r_1 - 2l}{r_2 + r_1 + 2l}\right) \end{array}\right\} \tag{6}$$

We define $s = r_2 + r_1$, $d = r_1 - r_2$ and $p = r_2 r_1$ as a auxiliary functions depending only on distances r_1 and r_2 of the particle to the end points of the segment.

The expression (6) reduce to :

$$U(P) = -\frac{G}{32l^2}\left\{12alsd^2 - 16al^3 s + \left[8l^2 a\left(s^2 - 2p\right) - 3as^2d^2 - 16l^4 a + 32bl^2\right]\ln\left(\frac{s+2l}{s-2l}\right)\right\} \quad (7)$$

(7) represent the gravitational potential generated by an inhomogeneous straight segment with a quadratic profile of density, this expression is our main result, to have more details and study about see $[7]$ Najid et al. The case of constant density $[1]$ is a particular situation of (7), if we put $a = 0$ and $b = \frac{M}{2l} = \lambda$. We obtain the expression (1) in $[1]$.

III- Dynamical study

We plane to study the dynamical behavior of a test particle, with unit mass, located at P in the field of the inhomogeneous straight segment.

$R(O, x, y, z)$ is the sidereal referential frame, with the cylindrical coordinates $(\rho,\ \theta,\ x)$ as in Fig.3.

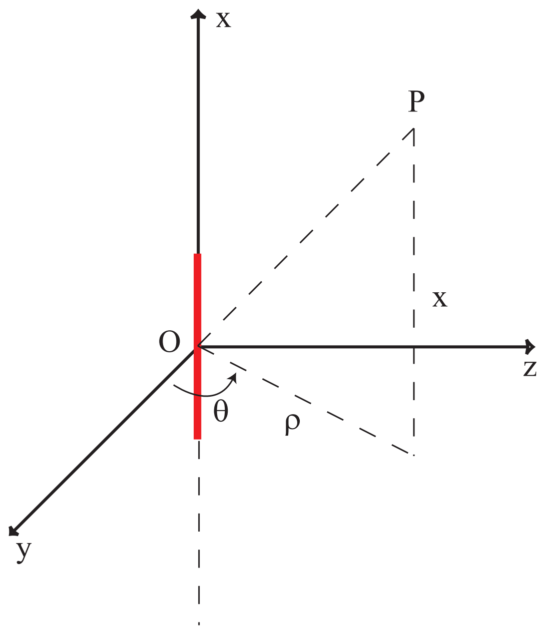

Fig. 3: *Sidereal referential and the cylindrical coordinates.*

The Lagrangian of the test particle is given by :

$$L = \frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{x}^2\right) - U(r_1, r_2)$$

where $r_1 = \sqrt{\rho^2 + (x+l)^2}$ and $r_2 = \sqrt{\rho^2 + (x-l)^2}$.

The conjugate momentum are :$P_\rho = \dot{\rho}$, $P_\theta = \rho^2\dot{\theta}$ and $P_x = \dot{x}$.

The Lagrange's equation corresponding to the coordinate ρ is

$$\frac{\partial L}{\partial \rho} = \rho\dot{\theta}^2 - \frac{\partial U(P)}{\partial \rho} = \ddot{\rho}$$

The differential equation of motion corresponding to ρ is given by

$$\ddot{\rho} = \rho\dot{\theta}^2 + \frac{G}{32l^2p}\left\{\begin{array}{l} 32al^2p\rho\ln\left(\frac{s+2l}{s-2l}\right) - 4al\rho s\left(3d^2+4l^2\right) \\ -\frac{4l\rho s}{s^2-4l^2}\left[8l^2a\left(s^2-2p\right) - 3as^2d^2 - 16l^4a + 32bl^2\right] \end{array}\right\} \tag{8}$$

The Lagrange's equation corresponding to the coordinate x is

$$\frac{\partial L}{\partial x} = -\frac{\partial U(P)}{\partial x} = \ddot{x}$$

The differential equation of motion corresponding to ρ is given by

$$\ddot{x} = \frac{Ga}{16l^2p}\left\{\begin{array}{l} 2l\left(xs-ld\right)\left(3d^2-4l^2\right) + 12lsd\left(ls-xd\right) + \\ \left[s\left(xs-ld\right)\left(8l^2-3d^2\right) - 8l^2x\left(s^2-2p\right) + 8l^3sd - 3s^2d\left(ls-xd\right)\right]\ln\left(\frac{s+2l}{s-2l}\right) \\ -\frac{2l\left(xs-ld\right)}{s^2-4l^2}\left[8l^2\left(s^2-2p\right) - 3s^2d^2 - 16l^4 + \frac{32bl^2}{a}\right] \end{array}\right\} \tag{9}$$

The Lagrange's equation corresponding to the coordinate θ is

$$\frac{\partial L}{\partial \theta} = 0$$

The differential equation of motion corresponding to θ is given by

$$\rho^2\dot{\theta} = \Lambda = cste \tag{10}$$

The case of homogeneous profile of density, $a = 0$ and $b = \lambda = \frac{M}{2l}$, lead to the equations

$$\ddot{\rho} = \frac{\Lambda^2}{\rho^3} - \frac{2\mu s\rho}{p\left(s^2-4l^2\right)}$$

$$\ddot{x} = -\frac{2\mu x}{sp}$$

We obtain the equation (3) as in $[1]$. In our case of inhomogeneous straight segment (8), (9) and (10) are strongly non linear and coupled. It need a deep numerical treatment. In fact, it is out of view to plane to work it out in an analytical way.

IV- Numerical integration

To have a deep insight about the dynamical behavior of the test particle in the field of the inhomogeneous straight segment, we have to solve (8), (9) and (10). In this system of differential equations the unknowns are ρ, θ and x. We derive some curves both, in the plan and in the space.

Fig.4, Fig.5 and Fig.6 give some orbits in the plan and in the space corresponding to different initial conditions. We notice, in a qualitative point of view, the existence of many behavior , we obtain the state :

- Collision,
- confined,
- not confined.

More analysis about the curves below are developed in $[7]$ Najid et al.

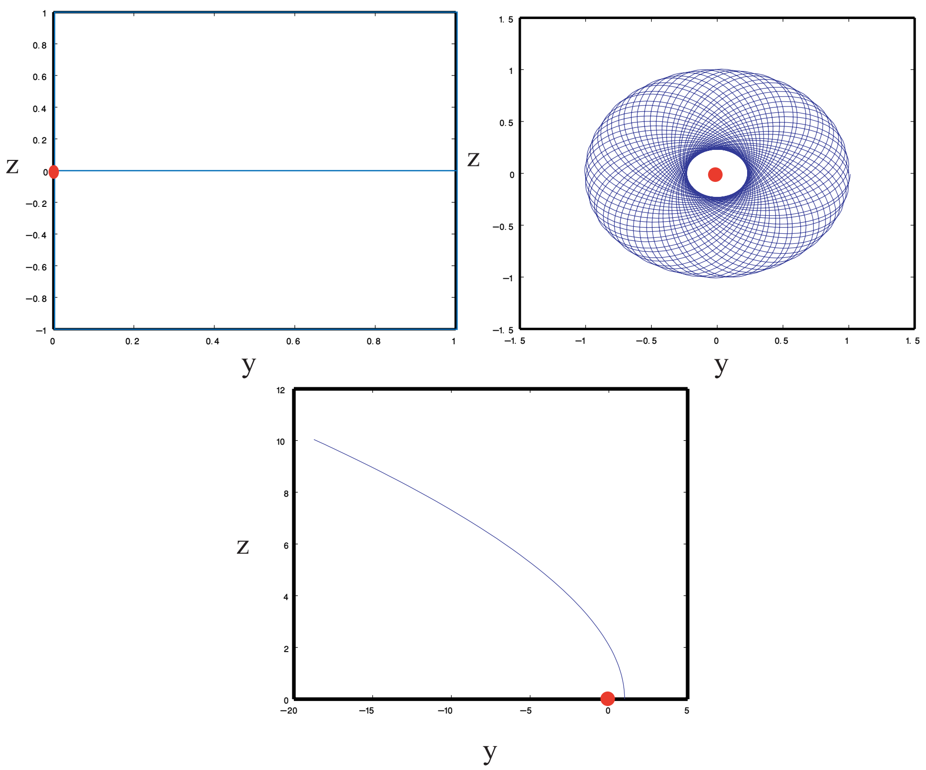

Fig. 4: *Trajectories in the plan* yz .

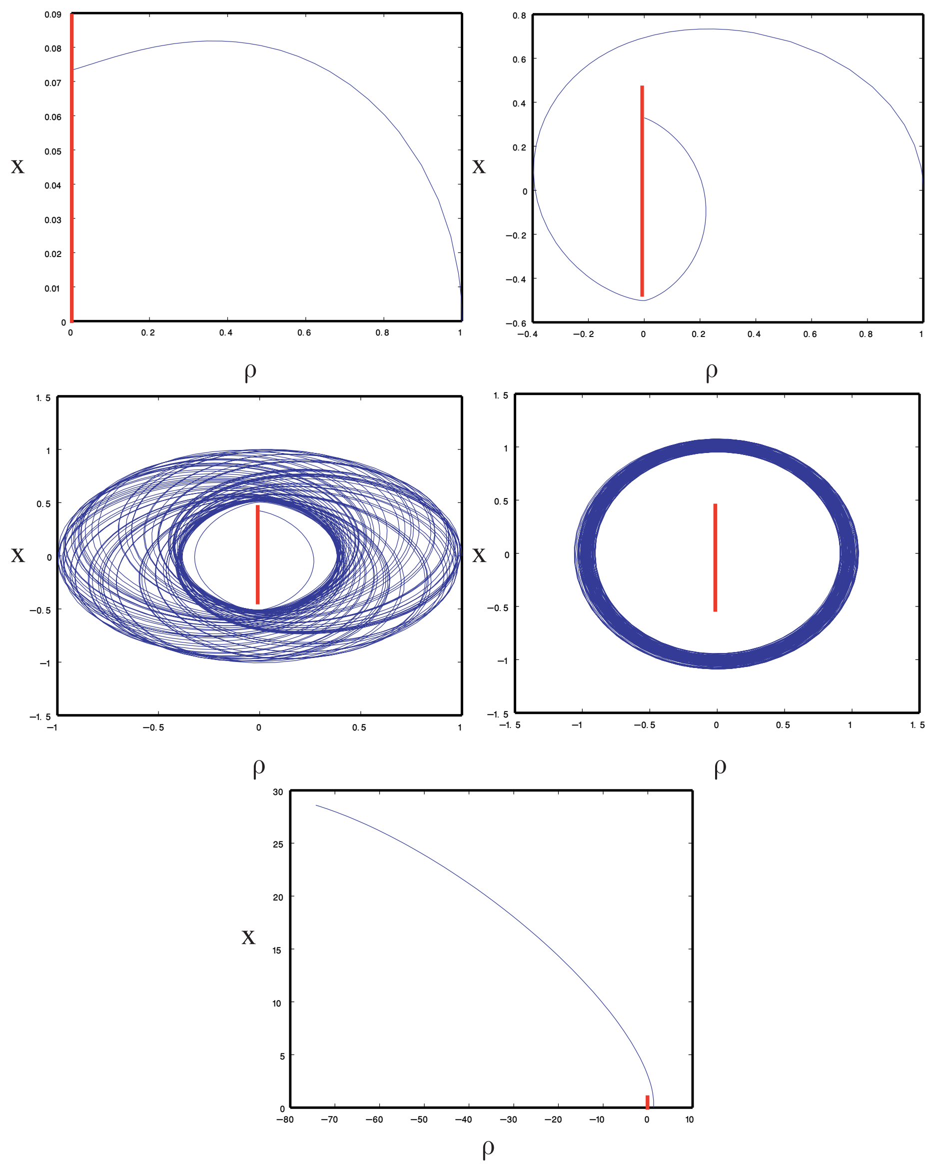

Fig. 5: *Trajectories in the plan $x\rho$.*

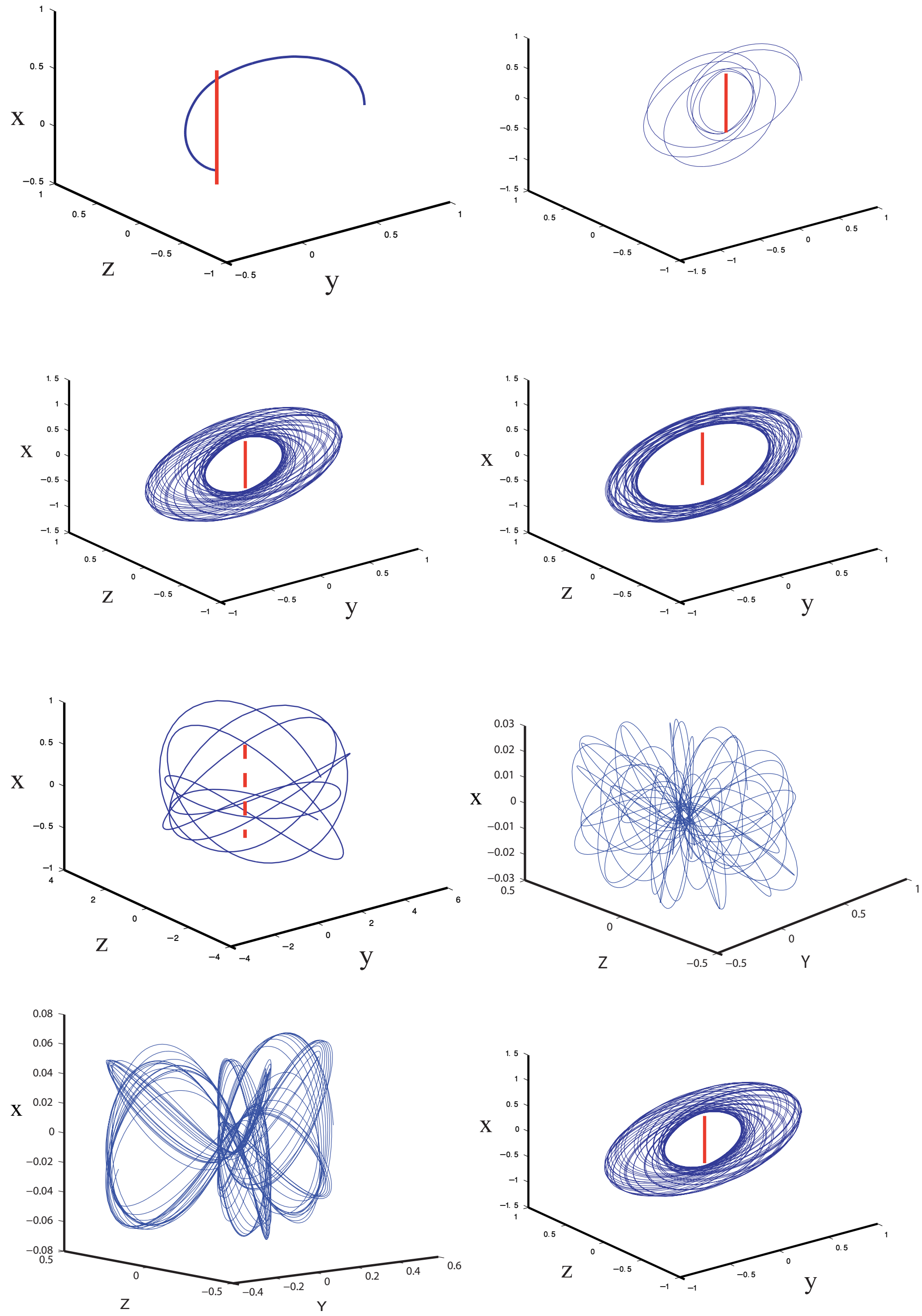

Fig. 6: *Trajectories in the space xyz.*

V- Conclusion

In this work, we established the analytical expression of the potential generated by a straight segment with a quadratic profile of its density. This potential model in an accurate manner celestial elongated bodies in the solar system. We derived some curves (trajectories) both in two and three dimensions. They gave an overview of the dynamical behavior of massless test particle. A deep study is fulfilled in [7] Najid et al. by using the poincaré surface of section. After the achievement of the dynamical behavior of a test particle in the field of that segment, fixed in space, we plane, in a next future, to study the case where the segment is in rotation.

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